International Journal of Humanities and Social Sciences
p-ISSN: 1694-2620
e-ISSN: 1694-2639
Vol. 8, No. 2, pp. 112-123, ©IJHSS

On a q-Analogue of the Non-central Whitney Numbers

Omar I. Cauntongan
Department of Natural Sciences and Mathematics
Mindanao State University-Maigo School of Arts and Trades
Maigo, Lanao del Norte 9206
Philippines
o.cauntongan@yahoo.com
o.cauntongan@gmail.com

Abstract
In this paper, a q-analogue of the noncentral Whitney numbers of both kinds are define in terms of horizontal generating functions. Some properties such as recurrence relations, explicit formula, generating functions, orthogonality and inverse relations are established. Matrix decomposition of these q-analogues is presented in an explicit and non-recursive form. Moreover, a q-analogue of the noncentral Dowling numbers and polynomials are defining and establish some of their properties.

Keywords: Whitney numbers, noncentral

Introduction

Stirling numbers was extensively studied by several mathematicians due to its applications in deferent field of discipline and its relatively interesting correlation properties with other branches of mathematics. The Whitney numbers is an extension of the classical Stir- ling numbers. Translated Whitney numbers by Belbachir and Bousbaa (2013), r-Whitney numbers by Cheon and Jung (2012) and noncental Whitney numbers by Mangontarum et al. (2014) are few of many extensions of the classical Whitney numbers.

On the other hand, Stirling numbers was extended into r-Stirling numbers by Broder (1984)) noncentral Stirling numbers by Koutras (1982) and (r; β)-Stirling numbers by Corcino (2012). To unify all extensions of Stirling-type and Whitney-type numbers, Hsu and Shiue (1998) introduced the unified generalized Stirling numbers in which all former extensions of these numbers were just a special case. Though some extensions are equivalent by proper choice of assignment of variables but their motivations and methodlogies in defining those numbers are different. Hence, more extensions of Stirling-type and Whitney-type numbers are still common interest of some mathematics researchers.

Recently, Mangontarum et.al (2014) defined a noncentral version of Whitney numbers parallel to the work of Koutras (1982) as follows:

$$(t|m)_n = \sum_{k=0}^{n} \widetilde{w}_{m,a}(n,k)(t-a)^k \quad \text{and} \tag{1}$$

$$(t-a)^n = \sum_{k=0}^{n} \widetilde{W}_{m,a}(n,k)(t|m)_k \tag{2}$$

where $(t|\alpha)_n = t(t-\alpha)(t-2\alpha)\cdots(t-(n-1)\alpha)$ and $(t|\alpha)_0 = 1$. We called $\widetilde{w}_{m,a}(n,k)$ and $\widetilde{W}_{m,a}(n,k)$ as *noncentral Whitney numbers of the first and second kinds* respectively. Further properties of these numbers were established in [17].

The study of a q-analogues and p,q-analogues of some well-known numbers and their properties has been the interest of several mathematicians of the 21^{st} century including Corcino [10, 11, 12], Conrad [7], Carlitz [5], Gould [15] and many others. The following are q-analogue of some special numbers such as x, $x!$ and $\binom{x}{k}$ are given by $[x]_q = \dfrac{1-q^x}{1-q}$, $[x]_q! = [1]_q[2]_q \ldots [x-1]_q[x]_q$ and $\begin{bmatrix} x \\ k \end{bmatrix}_q = \dfrac{[x]_q!}{[x-k]_q![k]_q!}$ respectively. Another important q-analogue identity is the known *q-binomial inversion formula*

$$f_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q g_k \Longleftrightarrow g_n = \sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q f_k. \tag{3}$$

A q-analogue of the product of two numbers a and b is given by

$$\begin{aligned}[ab]_q &= \frac{1-(q^b)^a}{1-q} \cdot \frac{1-q^b}{1-q^b} = \frac{1-(q^b)^a}{1-q^b} \cdot \frac{1-q^b}{1-q} \\ &= [a]_{q^b}[b]_q. \end{aligned} \tag{4}$$

Similarly, we can shown that $[ab]_q = [b]_{q^a}[a]_q$. The following are the type I and type II q-analogue of e^t respectively.

$$e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \quad \text{and} \quad \hat{e}_q(t) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{t^n}{[n]_q!}. \tag{5}$$

In this paper, a q-analogue of the noncentral Whitney numbers of the first and second kind in terms of horizontal generating functions are introduced. Some properties such as recurrence relations, explicit formula, generating functions, orthogonality and inverse relations are established. We give a matrix decomposition of these q-analogue in an explicit and non-recursive form. Moreover, a q-analogue of the noncentral Dowling numbers and polynomials are defined and some fundamental properties are established.

Noncentral Whitney Numbers of the First Kind

For convenience, let [t] = [t]q to denote the q-analogue of a real number t throughout this paper. Before we can define analogues of this study, let us first state the usual definition of a q-analogue of the falling factorial with increment n as;

$$([t|\alpha])_n = [t][t-\alpha]\cdots[t-(n-1)\alpha] = \prod_{k=0}^{n-1}[t-k\alpha], \tag{6}$$

where $([t|\alpha])_0 = 1$. We can easily verified $([t|\alpha])_n \to (t/\alpha)_n$ as $q \to 1$ where $(t|\alpha)_n = (t)(t-\alpha)\cdots(t-(n-1)\alpha)$.

Definition 2.1. A q-analogue $\widetilde{w}_{m,a}[n,j]_q$ of $\widetilde{w}_{m,a}(n,j)$ is defined via a horizontal generating function

$$([t|m])_n = \sum_{j=0}^{n} \widetilde{w}_{m,a}[n,j]_q[t-a]^j \tag{7}$$

where $\widetilde{w}_{m,a}[0,0]_q = 1$ and $\widetilde{w}_{m,a}[n,j]_q = 0$ for $j > n$.

Corollary 2.2. *The following identities hold:*

$$\sum_{j=0}^{n}\widetilde{w}_{m,a}[n,j]_q = ([a+1|m])_n \quad \textit{and} \tag{8}$$

$$\sum_{j=0}^{n}\widetilde{w}_{m,a}[n,j]_q[-1]^j = ([a-1|m])_n\,. \tag{9}$$

Proof. Replacing x by $a+1$ and $a-1$ in (7) yield (8) and (9) respectively. ❑

Theorem 2.3. *The following triangular recurrence relation holds:*

$$\widetilde{w}_{m,a}[n+1,j]_q = q^{a-mn}\widetilde{w}_{m,a}[n,j-1]_q + [a-nm]_q\widetilde{w}_{m,a}[n,j]_q \tag{10}$$

with $\widetilde{w}_{m,a}[n+1,0]_q = ([a|m])_{n+1}$.

Proof. Applying (7), we have

$$\begin{aligned}
\sum_{j=0}^{n+1}\widetilde{w}_{m,a}[n+1,j]_q[t-a]^j &= ([t|m])_n\left(q^{a-mn}[t-a]+[a-mn]\right)\\
&= q^{a-mn}\sum_{j=0}^{n}\widetilde{w}_{m,a}[n,j]_q[t-a]^j[t-a] + [a-mn]\sum_{j=0}^{n}\widetilde{w}_{m,a}[n,j]_q[t-a]^j\\
&= q^{a-mn}\sum_{j=0}^{n}\widetilde{w}_{m,a}[n,j]_q[t-a]^{j+1} + [a-mn]\sum_{j=0}^{n}\widetilde{w}_{m,a}[n,j]_q[t-a]^j\\
&= q^{a-mn}\sum_{j=0}^{n+1}\widetilde{w}_{m,a}[n,j-1]_q[t-a]^k + [a-mn]\sum_{j=0}^{n+1}\widetilde{w}_{m,a}[n,j]_q[t-a]^j\\
&= \sum_{k=0}^{n+1}\left(q^{a-mn}\widetilde{w}_{m,a}[n,j-1]_q + [a-mn]\widetilde{w}_{m,a}[n,j]_q\right)[t-a]^j.
\end{aligned}$$

Taking the coefficients of $[t\text{-}a]^1$ yields (10).

From (10), the following relations hold:

$$\widetilde{w}_{m,a}[n+1,j+1]_q = q^{a-mn}\widetilde{w}_{m,a}[n,j]_q + [a-nm]\widetilde{w}_{m,a}[n,j+1]_q; \tag{11}$$

$$\widetilde{w}_{m,a}[n,j]_q = \frac{\widetilde{w}_{m,a}[n+1,j+1]_q - [a-nm]\widetilde{w}_{m,a}[n,j+1]_q}{q^{a-mn}} \quad \text{and} \tag{12}$$

$$\widetilde{w}_{m,a}[n,j]_q = \frac{\widetilde{w}_{m,a}[n+1,j]_q - q^{a-mn}\widetilde{w}_{m,a}[n,j+1]_q}{[a-nm]}. \tag{13}$$

With these relations, we can establish the following theorem.

Theorem 2.4. *The following recursion formulas hold:*

$$\widetilde{w}_{m,a}[n+1,k+1]_q = \sum_{j=0}^{n} q^{a-mj}\left([a-m|m]\right)_{n-j}\widetilde{w}_{m,a}[j,k]_q \tag{14}$$

$$\widetilde{w}_{m,a}[n,k]_q = \sum_{j=0}^{n} \frac{(-1)^j[a-mn]^j\widetilde{w}_{m,a}[n+1,k+1+j]_q}{q^{(j+1)(a-mn)}} \quad \textit{and} \tag{15}$$

$$\widetilde{w}_{m,a}[n,k]_q = \sum_{j=0}^{n} \frac{(-1)^j q^{(j)(a-mn)}\widetilde{w}_{m,a}[n+1,k+1+j]_q}{[a-mn]^{j+1}}. \tag{16}$$

Proof. Successive application of (11), (12) and (13) yield (14), (15) and (17) respectively. □

Noncentral Whitney Numbers of the Second Kind

Definition 3.1. A q-analogue $\widetilde{W}_{m,a}[n,k]_q$ of $\widetilde{W}_{m,a}(n,k)$ is defined as follows:

$$[t-a]^n = \sum_{j=0}^{n} \widetilde{W}_{m,a}[n,j]_q([t|m])_j. \tag{17}$$

Theorem 3.2. *The following triangular recurrence relation holds:*

$$\widetilde{W}_{m,a}[n+1,j]_q = q^{m(j-1)-a}\widetilde{W}_{m,a}[n,j-1]_q + [mj-a]\widetilde{W}_{m,a}[n,j]_q. \tag{18}$$

Proof.

$$\begin{aligned}\sum_{j=0}^{n+1}\widetilde{W}_{m,a}[n+1,j]_q([t|m])_j &= [t-a]^n[t-a] = [(t-mj)+(mj-a)]_q\\ &=\sum_{k=0}^{n}\widetilde{W}_{m,a}[n,j]([t|m])_j\left(q^{mj-a}[t-mj]+[mj-a]\right)\\ &=\sum_{j=0}^{n}q^{mj-a}\widetilde{W}_{m,a}[n,j]_q([t|m])_{j+1}+\sum_{j=0}^{n}[mj-a]\widetilde{W}_{m,a}[n,k]_q([t|m])_k\\ &=\sum_{j=0}^{n+1}q^{m(j-1)-a}\widetilde{W}_{m,a}[n,j-1]_q([t|m])_k+\sum_{j=0}^{n+1}[mj-a]\widetilde{W}_{m,a}[n,j]_q([t|m])_k\\ &=\sum_{j=0}^{n+1}\left(q^{m(j-1)-a}\widetilde{W}_{m,a}[n,j-1]_q+[mk-a]\widetilde{W}_{m,a}[n,j]_q\right)([t|m])_k.\end{aligned}$$

Taking the coefficients of $([t|m])_j$ of both sides gives (18). □

The following relations can be deduced from (11), (12) and (13) using (18):

$$\widetilde{W}_{m,a}[n+1,j+1]_q = q^{m(j+1)-a}\widetilde{W}_{m,a}[n,j]_q+[m(j+1)-a]\widetilde{W}_{m,a}[n,j+1]_q; \tag{19}$$

$$\widetilde{W}_{m,a}[n,j]_q = \frac{\widetilde{W}_{m,a}[n+1,j+1]_q-[m(j+1)-a]\widetilde{W}_{m,a}[n,j+1]_q}{q^{mj-a}} \quad \text{and} \tag{20}$$

$$\widetilde{W}_{m,a}[n,j]_q = \frac{\widetilde{W}_{m,a}[n+1,j]_q-q^{m(j-1)-a}\widetilde{W}_{m,a}[n,j+1]_q}{[mj-a]}. \tag{21}$$

Using the above relations, we can establish following theorem.

Theorem 3.3. *The following recurrence relations hold:*

$$\widetilde{W}_{m,a}[n+1,k+1]_q = q^{mk-a}\sum_{j=k}^{n}[m(k+1)-a]^{n-j}\widetilde{W}_{m,a}[j,k]_q \tag{22}$$

$$\widetilde{W}_{m,a}[n,k]_q = \sum_{j=0}^{n-k}\frac{(-1)^j\left([m(k+1)-a|-m]\right)_j\widetilde{W}_{m,a}[n+1,k+1+j]_q}{q^{(j+1)(mk-a)+\binom{j+1}{2}m}} \quad \textit{and} \tag{23}$$

$$\widetilde{W}_{m,a}[n,k]_q = \sum_{j=0}^{k}\frac{(-1)^j q^{j(m(k-1)-a)-\binom{j}{2}m}\widetilde{W}_{m,a}[n+1,k-j]_q}{([mk-a|m])_{j+1}}. \tag{24}$$

Proof. Successive application of (19), (20) and (21) yield (22), (23) and (24) respectively. □

Theorem 3.4. *The following explicit formula holds:*

$$\widetilde{W}_{m,a}[n,k]_q = \frac{1}{[k]_{q^m}![m]^k}\sum_{j=0}^{k}(-1)^{k-j}q^{m\binom{k-j}{2}}\begin{bmatrix}k\\ j\end{bmatrix}_{q^m}[mj-a]^n. \tag{25}$$

Proof. Replacing t by mk in (17) gives

$$[mk-a]^n = \sum_{j=0}^{n} \widetilde{W}_{m,a}[n,j]_q([mk|m])_j$$
$$= \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_{q^m} \left\{ \frac{\widetilde{W}_{m,a}[n,j]_q([mk|m])_j}{\begin{bmatrix} k \\ j \end{bmatrix}_{q^m}} \right\}$$

Using the q-binomial inversion formula (3), we obtain

$$\frac{\widetilde{W}_{m,a}[n,k]_q([mk|m])_k}{\begin{bmatrix} k \\ k \end{bmatrix}_{q^m}} = \sum_{j=0}^{k} (-1)^{k-j} q^{m\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^m} [mj-a]^n.$$

Applying (4) and simplifying above give us (25). □

Theorem 3.5. *The following exponential generating function holds:*

$$\sum_{n=i}^{\infty} \widetilde{W}_{m,a}[n,i]_q \frac{[z]^n}{[n]!} = \frac{1}{[i]_{q^m}![m]^i} \sum_{j=0}^{i} (-1)^{i-j} q^{m\binom{i-j}{2}} \begin{bmatrix} i \\ j \end{bmatrix}_{q^m} e_q\left([mj-a]\,[z]\right). \tag{26}$$

Proof. Using the explicit formula in (25) gives

$$\sum_{n=i}^{\infty} \widetilde{W}_{m,a}[n,i]_q \frac{[z]^n}{[n]!} = \sum_{n=i}^{\infty} \frac{1}{[i]_{q^m}![m]^i} \sum_{j=0}^{i} (-1)^{i-j} q^{m\binom{i-j}{2}} \begin{bmatrix} i \\ j \end{bmatrix}_{q^m} [mj-a]^n \frac{[z]^n}{[n]!}$$
$$= \frac{1}{[i]_{q^m}![m]^i} \sum_{j=0}^{i} (-1)^{i-j} q^{m\binom{i-j}{2}} \begin{bmatrix} i \\ j \end{bmatrix}_{q^m} \sum_{n=i}^{\infty} \frac{[mj-a]^n [z]^n}{[n]!}.$$

Applying (5) gives (26). □

Theorem 3.6. *Let $k, m, n, p > 0$ integers and any real numbera. Then the following orthogonality relation holds:*

$$\sum_{k=j}^{n} \widetilde{W}_{m,a}[n,k]_q \widetilde{w}_{m,a}[k,p]_q = \sum_{k=j}^{n} \widetilde{w}_{m,a}[n,k]_q \widetilde{W}_{m,a}[k,p]_q = \begin{cases} 0, & if\ p \neq n \\ 1, & if\ p = n \end{cases} = \delta_{np} \tag{27}$$

where δ_{np} is a Kronecker delta.

Proof. Using (7) and (17) gives

$$[t-a]^n = \sum_{k=0}^{n} \widetilde{W}_{m,a}[n,k]_q \sum_{p=0}^{n} \widetilde{w}_{m,a}[k,p]_q [t-a]^p$$
$$= \sum_{p=0}^{n} \left\{ \sum_{k=p}^{n} \widetilde{W}_{m,a}[n,k]_q \widetilde{w}_{m,a}[k,p]_q \right\} [t-a]^p.$$

Comparing the coefficients of $[t-a]^p$ gives

$$\sum_{k=p}^{n} \widetilde{W}_{m,a}[n,k]_q \widetilde{w}_{m,a}[k,p]_q = \begin{cases} 0, & if\ p \neq n \\ 1, & if\ p = n \end{cases} = \delta_{np}.$$

Similarly, we can easily prove that

$$\sum_{k=p}^{n} \widetilde{w}_{m,a}(n,k) \widetilde{W}_{m,a}(k,p) = \delta_{np}. \qquad \square$$

Theorem 3.7. *Let $k,n,m > 0$ integers and any real numbers a. Then the following inverse relation holds:*

$$f_n = \sum_{k=0}^{n} \widetilde{w}_{m,a}[n,k]_q g_k \Longleftrightarrow g_n = \sum_{k=0}^{n} \widetilde{W}_{m,a}[n,k]_q f_k. \tag{28}$$

Proof. If the condition

$$f_n = \sum_{k=0}^{n} \widetilde{w}_{m,a}[n,k]_q g_k$$

is true, then

$$\begin{aligned} \sum_{k=0}^{n} \widetilde{W}_{m,a}[n,k]_q f_k &= \sum_{k=0}^{n} \widetilde{W}_{m,a}[n,k]_q \sum_{p=0}^{k} \widetilde{w}_{m,a}[n,p]_q g_p \\ &= \sum_{m=0}^{n} \left\{ \sum_{k=p}^{n} \widetilde{W}_{m,a}[n,k]_q \widetilde{w}_{m,a}[n,p]_q \right\} g_p. \end{aligned}$$

By (27), we have

$$\begin{aligned} \sum_{k=0}^{n} \widetilde{W}_{m,a}[n,k]_q f_k &= \sum_{p=0}^{n} \delta_{pn} g_p = \delta_{nn} g_n \\ &= g_n. \end{aligned}$$

Similarly, we can show the converse using similar argument.

Matrix Representation of q-Whitney Numbers

In the paper of Pan (2012) the matrix representation of the unified generalized Stirling numbers was defined and established a matrix decomposition of this numbers in an explicit and non-recursive form. Parallel to this work, we will establish matrix decompositions for the q-analogue of noncentral Whitney numbers. From (7) and (17), we have the matrix representation of the q-analogue of noncentral Whitney numbers of the first and second kind respectively as

$$\mathcal{V}_m[t] = \mathcal{M}^q_{m,a} \mathcal{V}_0[t-a] \tag{29}$$

and

$$\mathcal{V}_0[t-a] = \mathcal{N}^q_{m,a}\mathcal{V}_m[t] \tag{30}$$

where

$$\mathcal{V}_\alpha[t] = (1, [t], ([t|\alpha])_2, ([t|\alpha])_3, \cdots, ([t|\alpha])_n, \cdots)^T.$$

For convenience, we write (29) and (30) explicitly as

$$\begin{pmatrix} 1 \\ [t] \\ ([t|m])_2 \\ ([t|m])_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} \widetilde{w}_{m,a}[0,0]_q & 0 & 0 & 0 & \cdots \\ \widetilde{w}_{m,a}[1,0]_q & \widetilde{w}_{m,a}[1,1]_q & 0 & 0 & \cdots \\ \widetilde{w}_{m,a}[2,0]_q & \widetilde{w}_{m,a}[2,1]_q & \widetilde{w}_{m,a}[2,2]_q & 0 & \cdots \\ \widetilde{w}_{m,a}[3,0]_q & \widetilde{w}_{m,a}[3,1]_q & \widetilde{w}_{m,a}[3,2]_q & \widetilde{w}_{m,a}[3,3]_q & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ ([t-a|0])_1 \\ ([t-a|0])_2 \\ ([t-a|0])_3 \\ \vdots \end{pmatrix}$$

and

$$\begin{pmatrix} 1 \\ ([t-a|0])_1 \\ ([t-a|0])_2 \\ ([t-a|0])_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} \widetilde{W}_{m,a}[0,0]_q & 0 & 0 & 0 & 0 & \cdots \\ \widetilde{W}_{m,a}[1,0]_q & \widetilde{W}_{m,a}[1,1]_q & 0 & 0 & 0 & \cdots \\ \widetilde{W}_{m,a}[2,0]_q & \widetilde{W}_{m,a}[2,1]_q & \widetilde{W}_{m,a}[2,2]_q & 0 & 0 & \cdots \\ \widetilde{W}_{m,a}[3,0]_q & \widetilde{W}_{m,a}[3,1]_q & \widetilde{W}_{m,a}[3,2]_q & \widetilde{W}_{m,a}[3,3]_q & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ [t] \\ ([t|m])_2 \\ ([t|m])_3 \\ \vdots \end{pmatrix}$$

respectively.

Remark 3.8. By orthogonality relation in (27), we have

$$\mathcal{M}^q_{m,a} \cdot \mathcal{N}^q_{m,a} = \mathcal{N}^q_{m,a} \cdot \mathcal{M}^q_{m,a} = \mathcal{E}, \tag{31}$$

where E is an in_nite-dimensional identity matrix. Hence,Mq m;a = Inv^{Nq} m;a , where Inv^{Nq} m;a is the inverse of N^q m;a and vice versa.

Theorem 3.9. The following matrix decomposition formulas hold:

$$\mathcal{M}^q_{m,a} = \mathcal{M}^q_{m,0} \cdot \mathcal{M}^q_{0,a} \quad and \tag{32}$$

$$\mathcal{N}^q_{m,a} = \mathcal{N}^q_{0,a} \cdot \mathcal{N}^q_{m,0}. \tag{33}$$

Proof. By (29), $\mathcal{V}_m[x] = \mathcal{M}^q_{m,a}\mathcal{V}_0[x-a]$. Note that when the increment $m=0$ and $a \neq 0$, we have $\mathcal{V}_0[x] = \mathcal{M}^q_{0,a}\mathcal{V}_0[x-a]$. On the otherhand when $m \neq 0$ and $a = 0$, we have $\mathcal{V}_m[x] = \mathcal{M}^q_{m,0}\mathcal{V}_0[x]$. Since $\mathcal{V}_m[x] = \mathcal{M}^q_{m,a}\mathcal{V}_0[x-a]$ and $\mathcal{V}_m[x] = \mathcal{M}^q_{m,0}\mathcal{V}_0[x]$, applying transitive property yield $\mathcal{V}_m[x] = \mathcal{M}^q_{m,a}\mathcal{V}_0[x-a] = \mathcal{M}^q_{m,0}\mathcal{M}^q_{0,a}\mathcal{V}_0[x-a]$. It follows that $\mathcal{M}^q_{m,a}\mathcal{V}_0[x-a] - \mathcal{M}^q_{m,0}\mathcal{M}^q_{0,a}\mathcal{V}_0[x-a] = \mathbf{0}$ where $\mathbf{0}$ denotes an infinite dimensional zero matrix. Hence, $(\mathcal{M}^q_{m,a} - \mathcal{M}^q_{m,0}\mathcal{M}^q_{0,a})\mathcal{V}_0[x-a] = \mathbf{0}$. Since x is an arbitrary real or complex number and $\mathcal{V}_0[x-a]$ is a nonzero vector, it yields (32). It is easy to show (33) using similar method and it is left as an exercise. □

Example 3.10. Consider a 3×3 of $\mathcal{M}^q_{m,a}$, $\mathcal{M}^q_{0,a}$, $\mathcal{M}^q_{m,0}$ $\mathcal{N}^q_{m,a}$, $\mathcal{N}^q_{0,a}$ and $\mathcal{N}^q_{m,0}$. Then we have

$$\mathcal{M}^q_{m,0} \cdot \mathcal{M}^q_{0,a} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & [-m] & q^{-m} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ [a] & q^a & 0 \\ [a]^2 & 2[a]q^a & q^{2a} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ [a] & q^a & 0 \\ [a-m][a] & q^{a-m}[a] + q^a[a-m] & q^{2a-m} \end{pmatrix} = \mathcal{N}$$

Also that

$$\mathcal{N}^q_{0,a} \cdot \mathcal{N}^q_{m,0} = \begin{pmatrix} 1 & 0 & 0 \\ [-a] & q^{-a} & 0 \\ [-a]^2 & 2[-a]q^{-a} & q^{-2a} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & [m] & q^m \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ [-a] & q^{-a} & 0 \\ [-a]^2 & [-a]q^{-a} + [m-a]q^{-a} & q^{m-2a} \end{pmatrix} = \mathcal{N}^q_m$$

We can easily verify our example above by assigning an specific value for m and a. Clearly,

$$\mathcal{M}^q_{m,a} \cdot \mathcal{N}^q_{m,a} = \mathcal{N}^q_{m,a} \cdot \mathcal{M}^q_{m,a} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Noncentral q-Dowling Numbers

Dowling (1973) defined a class of geometric lattices based on finite groups. This is called Dowling numbers. Mangontarum et al. (2016) defined the noncentral and translated Dowling numbers and established some of its properties. A q-analogue of the noncentral Dowling numbers is introduce and obtain some combinatorial properties of it such as exponential generating function and Dobinski-Type formula.

Definition 4.1. The noncentral q-Dowling polynomials denoted by Dα[n; x]q, is defined as

$$\widetilde{D}_{m,a}[n;x]_q = \sum_{k=0}^{n} \widetilde{W}_{m,a}[n,k]_q x^k. \tag{34}$$

When $x = 1$ we get

$$\widetilde{D}_{m,a}[n]_q = \sum_{k=0}^{n} \widetilde{W}_{m,a}[n,k]_q \tag{35}$$

and is called translated q-Dowling numbers.

Theorem 4.2. *The polynomials $\widetilde{D}_{m,a}[n;x]_q$ satisfy the following exponential generating functions*

$$\widetilde{D}_{m,a}[n;x]_q = \hat{e}_{q^a}\left(-x/[m]\right) \sum_{i=0}^{\infty} \frac{e_q\left([mi-a]\,[z]\right)}{[i]_{q^a}!} \left(x/[m]\right)^i \tag{36}$$

and

$$\widetilde{D}_{m,a}[n]_q = \hat{e}_{q^a}\left(-/[m]\right) \sum_{i=0}^{\infty} \frac{e_q\left([mi-a]\,[z]\right)}{[i]_{q^a}!} \left(1/[m]\right)^i \tag{37}$$

Proof.

$$\begin{aligned}
\sum_{n\geq k}\widetilde{D}_{a}[n;x]_q\frac{[z]^n}{[n]_q!} &= \sum_{n\geq k}\left(\sum_{k=0}^{n}\widetilde{W}_{m,a}[n,k]_q x^k\right)\frac{[z]^n}{[n]!}\\
&= \sum_{k=0}^{n}\left(\sum_{n\geq k}\widetilde{W}_{m,a}[n,k]_q\frac{[z]^n}{[n]!}\right)x^k\\
&= \sum_{k=0}^{n}\frac{1}{[k]_{q^m}![m]^k}\sum_{j=0}^{k}(-1)^{k-j}q^{m\binom{k-j}{2}}\begin{bmatrix}k\\ j\end{bmatrix}_{q^m} e_q\left([mj-a]\,[z]\right)x^k\\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^j q^{m\binom{j}{2}}\begin{bmatrix}k\\ k-j\end{bmatrix}_{q^m} e_q\left([m(k-j)-a]\,[z]\right)x^k}{[k]_{q^m}![m]^k}\\
&= \sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{(-1)^j q^{m\binom{j}{2}} e_q\left([m(k-j)-a]\,[z]\right)x^k}{[k-j]_{q^m}![j]_{q^m}![m]^k}\\
&= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^j q^{m\binom{j}{2}} e_q\left([mi-a]\,[z]\right)x^{i+j}}{[i]_{q^m}![j]_{q^m}![m]^{i+j}}\\
&= \sum_{j=0}^{\infty}\frac{(-x)^j q^{m\binom{j}{2}}}{[m]^j[j]_{q^m}!}\cdot\sum_{i=0}^{\infty}\frac{e_q\left([mi-a]\,[z]\right)x^i}{[i]_{q^m}![m]_q^i}.
\end{aligned}$$

Simplifying above gives (36). Setting $x = 1$ yields (37). □

Remark 4.3. When $\alpha = 1$, we recover the the q-Bell Polynomials. That is

$$\widetilde{D}_1[n;x]_q = B_n[x]_q = \sum_{k=0}^{n} S_q(n,k)x^k. \tag{38}$$

Setting $x = 1$, we get $B_n[1]_q = B_q[n]$.

Theorem 4.4. *The polynomials $\widetilde{D}_{m,a}[n;x]_q$ satisfy the following explicit formula*

$$\widetilde{D}_{m,a}[n;x]_q = \hat{e}_{q^m}\left(-x/[m]\right)\sum_{i=0}^{\infty}\frac{[mi-a]^n}{[i]_{q^m}!}\left(x/[m]_q\right)^i. \tag{39}$$

When $x = 1$, we have

$$\widetilde{D}_{m,a}[n]_q = \hat{e}_{q^m}\left(-1/[m]\right)\sum_{i=0}^{\infty}\frac{[mi-a]^n}{[i]_{q^m}!}\left(1/[m]_q\right)^i. \tag{40}$$

and is called explicit formula for noncentral q-Dowling numbers.

Proof.

$$
\begin{aligned}
\widetilde{D}_a[n;x]_q &= \sum_{k=0}^{n} \widetilde{W}_{m,a}[n,k]_q x^k \\
&= \sum_{k=0}^{n} \frac{1}{[k]_{q^m}![m]^k} \sum_{j=0}^{k} (-1)^{k-j} q^{m\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^m} [mj-a]^n x^k \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^j q^{m\binom{j}{2}} \begin{bmatrix} k \\ k-j \end{bmatrix}_{q^m} [m(k-j)-a]_q^n x^k}{[k]_{q^m}![m]^k} \\
&= \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \frac{(-1)^j q^{m\binom{j}{2}} [m(k-j)-a]^n x^k}{[k-j]_{q^m}![j]_{q^m}![m]^k} \\
&= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^j q^{m\binom{j}{2}} [mi-a]^n x^{i+j}}{[i]_{q^m}![j]_{q^m}![m]_q^{i+j}} \\
&= \sum_{j=0}^{\infty} \frac{(-x)^j q^{m\binom{j}{2}}}{[m]^j [j]_{q^m}!} \cdot \sum_{i=0}^{\infty} \frac{[mi-a]^n x^i}{[i]_{q^m}![m]^i}.
\end{aligned}
$$

Simplifying above gives (39). Also that setting $x = 1$ yields (40).

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The Author

Omar Ibrahim Cauntongan born at Poblacion Munai Lanao del Norte Philippines on April 28, 1986. He finished both B.S. and M.S. in Mathematics at Mindanao State University-Main Campus, Marawi City Philippines. He is currently pursuing Ph.D. in Mathematics under DOST ASTHRD Scholarship at Mindanao State University-Iligan Institute of Technology, Iligan City Philippines. He is a faculty member of Department of Natural Sciences and Mathematics, Mindanao State University-Maigo School of Arts and Trades where he taught basic mathematics, calculus, ordinary differential equations and linear algebra. He wrote 8 researches related to Stirling-type and Whitney-type numbers. Four of his eight researches are under review for publication in some international peer-reviewed journals. Prof. Cauntongan is a member of Mathematical Society of the Philippines, DOST-SEI Scholar Society and a licensed secondary mathematics teacher. He presented some papers in national and international conferences namely: 1st International Multidisciplinary Research Conference 2016 of the Alliance of Teachers for Innovative Education and Excellence, Inc., 2015 MSP National Conference, 2015 MSP Cebu Research Congress and Convention and 1st In-House Research Conference of MSU-Marawi City. Recently, he published one of his researches entitled "The Noncentral Version of the Whitney Numbers: A Comprehensive Study" in which he co-authored Prof. Mahid M. Mangontarum and Prof. Amila P. Macodi-Ringia at International Journal of Mathematics and Mathematical Sciences (Scopus Indexed Journal).